OPERATIONAL RISK MODELS AND MAXIMUM LIKELIHOOD
ESTIMATION ERROR FOR SMALL SAMPLE-SIZES

PAUL LARSEN

Abstract. Operational risk models commonly employ maximum likelihood estimation (MLE) to fit loss data to heavy-tailed distributions. Yet several desirable properties of MLE (e.g. asymptotic normality) are generally valid only for large sample-sizes, a situation rarely encountered in operational risk. We study MLE in operational risk models for small sample-sizes across a range of loss severity distributions. We apply these results to assess (1) the approximation of parameter confidence intervals by asymptotic normality, and (2) value-at-risk (VaR) stability as a function of sample-size. Finally, we discuss implications for operational risk modeling.

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1. Introduction

Maximum likelihood estimation (MLE) is a— if not the—standard method for fitting parametric distributions in operational risk models [AG13]. Its widespread use is due in large part to properties that hold as the sample-size of loss data goes to infinity, namely, that MLE is a consistent, asymptotically normal, and asymptotically efficient estimator. We focus here on the second property, asymptotic normality of MLE, and how this property relates operational value-at-risk (OpVaR) models. Informally, asymptotic normality of MLE means that the estimated parameters will be normally distributed about the true parameters with variance going to zero as the sample-size tends to infinity (greater detail will be given in Section 2).
The assumption of sample-sizes approaching infinity, however, is hard to justify in operational risk, where capital estimates are driven by large and rare loss events, thus making the asymptotic nature of asymptotic normality ground for concern in OpVaR models. The situation is summarized in [EKM97], p. 318: “[A]lthough we have reliable numerical procedures for finding the MLE . . ., we are less certain about its properties, especially in the small sample case.”

The challenge of small sample-sizes for MLE estimation is exacerbated by the recent regulatory trend toward stand-alone OpVaR models. Rather than calculating a single OpVaR for all of a given bank’s legal entities, and then sub-allocating this capital figure to the legal entities, local regulators are beginning to ask for OpVaR models that calculate on the level of a given legal entity (or cluster of entities in a country). Hence the problem of fitting a model to a relatively small number of high losses across a bank has been made more severe by effectively carving the bank—and its loss data—into even smaller pieces.

While it is interesting to know how—if at all—asymptotic normality holds for heavy-tailed distributions, our interests are more practice-oriented, namely to

1. assess the normal approximation for parameter fitting confidence intervals,
2. translate MLE error into OpVaR error.

The first point refers to using asymptotic normality to determine confidence intervals for parameter estimates. Of course, these estimates are only reliable insofar as the parameter error is distributed as predicted by asymptotic normality. The second goal amounts to estimating error in high quantiles of the loss distribution. For the translation from parameter error to OpVaR error, we give an extension of the single-loss approximation of [BK05] for spliced distributions.

The OpVaR stability results can be used to create a modeling framework for both choosing a heavy-tailed distributions in a “small-data” environment of operational risk, and gauging how many losses are needed for a stable OpVaR model. Having a handle on how many large losses are required provides a key component to making other modeling decisions, such as

- What granularity is appropriate for the business line/event type composition of parametric tails?
- In the case of sliced severity distributions (the case of interest here), what are appropriate thresholds for the tail?
- If external loss data are filtered by geographical criteria, how granular of a regional division is appropriate?

While this paper addresses operational risk, the main problem of estimating errors resulting from MLE fitting of heavy-tailed distributions is much more general. The heavy-tailed distributions we consider (Pareto, lognormal, log-logistic and generalized Beta of the second kind) arise in many other contexts besides operational risk, such as data networks, market models and insurance [Res07]. Indeed, the type of OpVaR model considered here is commonly called the “actuarial approach,” and is characterized by modeling frequencies and severities of losses separately.

We now outline how our results relate to existing literature and practice. The use of asymptotic normality to estimate MLE error is common in practice, and is also used in [CMAU09] to estimate parameter confidence intervals. Possible shortcomings of this approximation are mentioned in [PS14].

To estimate OpVaR error we combine parametric bootstrapping and a variant of the single-loss OpVaR approximation for spliced severity distributions. There
are several benefits to this combination. In contrast to running a Monte Carlo loss simulation for each fitted parameter (as in e.g. [DP07]), the single-loss approximation is computationally efficient. A further possible extension would be to develop a propagation of error formula, which could give relative error about the true OpVaR (see e.g. [CMAU09]; note that the approximation employed there assumes a large sample-size). But as we will see in Section 3, even when the fitted parameters appear normally distributed about the true parameters, the resulting OpVaR statistics are not in general normally distributed. Hence a single-number from such a propagation error formula would tell only a small part of the story. Instead, we give a range of confidence bands about the true OpVaR. The non-spliced version of the single-loss approximation was used in [CMAU09] to quantify MLE error for one particular lognormal distribution, and in [OC12] for lognormal and loggamma distributions.

Other methods to quantifying the stability of MLE for OpVaR exist in the literature and in practice, the least rigorous of which is to add some artificial large loss to the data set and refit the data with MLE. A more systematic variant is to draw the artificial loss from a high quantile of the severity distribution. Both of these approaches aim at estimating how much OpVaR can increase in case of new, large losses. But the stability of OpVaR models with respect to new small losses is also important [OC12]. Parametric bootstrapping encapsulates both trends in a systematic framework. While we have chosen a range of loss data and distributions for our results in the hope that the conclusions can be directly applied to OpVaR modeling, our principal aim is to present a framework for developing and validating MLE severity calibration within OpVaR models in the face of small sample-sizes.

Per definition, parametric bootstrapping is compatible with one of the key assumptions of MLE, namely that data are independent and identically distributed, since the bootstrapped data samples are drawn independently from a single “true” distribution. That these assumptions hold for actual loss data has been questioned, and the resulting impact for loss data that are not independent and identically distributed on MLE is a focus of [OC12]. Operational risk literature is not unanimous on this question, but the regular assessments of the ORX loss data consortium offer general backing for this assumption [CA08, Ana12]. Besides the assumption of large sample-sizes, we investigate assumptions on distribution being fitted. It is, however, well-known that MLE can exhibit asymptotic normality even when the assumptions of the theory are not met (see e.g. [Smi85]). Loosely put, the assumptions are necessary to prove the result, but not necessarily for the result to hold. For three of the distributions (Pareto, lognormal, and log-logistic), we see high levels of agreement between theory and simulation, even for very small sample-sizes. The assumptions of asymptotic normality for the generalized Beta distribution of the second kind are difficult to verify.

We keep the standard MLE assumption that we know the “true” underlying distribution. Robustness of a fitting method to misspecified models (i.e. selecting the wrong “true” distribution) is a topic of active research; see e.g. [ErS08, OC12].

Relating theoretical concerns to practice requires representative loss data. We use loss data from the recent ORX Association OpVaR stability study, in which participants were given loss data sets for 12 units of measure (UOM) consisting of anonymized loss data from member banks. Four UOMs were selected for this paper that spanned a range of loss profiles. Due to space considerations, analyses
are given here for only one UOM (UOM1), with full results for this UOM and three others given in a separate appendix. [Lar15]

We now review the remainder of this paper. In Section 2 we present background material on OpVaR models, with special attention to spliced severity distributions, which are common in practice but have generally received less attention in the literature. We next describe MLE and asymptotic normality, and conclude the section with an overview of the distributions under consideration (Pareto, Weibull, lognormal, log-logistic, and generalized Beta of the second kind). In particular, we consider how these distributions do—and do not—fulfil the usual assumptions for asymptotic normality. Note that each of these distributions is put forth in literature as candidates for OpVaR severity distributions, although the current work seems to be the first time in the OR context that their theoretical properties under MLE are considered. We conclude this section by describing our simulations in detail and giving an example of asymptotic normality in action for one loss data source. In addition to graphical comparisons of theory and simulation, we also study confidence bands for parameter estimation derived from asymptotic normality with those from simulation.

Our main stability results are presented in Section 3 where we develop the single-loss-approximation to OpVaR for spliced distributions and translate bootstrapped parameters for one loss-data source into OpVaR error analyses. In Section 4 we relate the results of the previous sections to modeling decisions, and discuss directions for future investigation. Analogous results for three additional loss data sets are given in [Lar15].

These analyses lead to two main conclusions regarding the use of asymptotic normality to approximate parameter confidence intervals and OpVaR stability:

(1) Asymptotic normality provides a good approximation to parameter confidence intervals even for small sample sizes, with the exceptions of the Weibull and Generalized Beta Distribution of the Second Kind (GB2).

(2) Different severity distributions require substantially divergent sample-sizes to achieve a comparable level of OpVaR stability. For example, the flexibility of the GB2 distribution comes at a high price for stability.

To conclude this section, we mention a few technical points. For the probability distributions considered below, different sources give different names for parameters, and sometimes the distribution functions themselves vary from source to source. We give precise definitions in the next section, and generally follow parameter naming conventions as in the R packages below that were used for our analyses.

The statistical analyses, graphics and typesetting were all done via the statistical software R [R C14]. The R packages beyond the ones in the base set-up are fitdistrplus [DMPDD13], GB2 [GN14], ggplot2 [Wic09], neldermead [BB14], VGAM [Yee14], and Sweave [Lei02, Lei03].

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2. **OpVaR, asymptotic normality of MLE and heavy-tailed distributions**

2.1. **OpVaR basics.** A very common method for calculation OpVaR is the *loss distribution approach* (LDA). This method involves explicit modeling of a bank’s operational losses over a one-year time horizon to a high confidence level (e.g. the 99.9% quantile for Regulatory Capital).

The loss distributions we consider fall under the actuarial-approach, which means that the frequency and severity of losses are calibrated separately, and then combined for the full loss distribution, either by Monte Carlo simulation, Fast-Fourier-Transform, or another method such as the single-loss approximation discussed in Section 3. We focus on the severity modeling for several reasons. As mentioned in the introduction, OpVaR figures tend to be driven by rare large losses; this translates into the common wisdom that OpVaR is more sensitive to severity modeling than frequency modeling. Moreover, frequency distributions are generally calibrated to all loss data, while severity distributions are often calibrated only to losses above a certain threshold. Hence small sample sizes are a rare concern on the frequency side. One final reason for focusing on severity modeling comes from our interest in model selection, namely, which heavy tailed distributions perform best with small sample-sizes. For modeling loss frequencies, the choice of distribution family is much more restricted: banks typically use only two types of distributions, Poisson or negative-binomial [AG13].

On the severity side, there is a range of common practice about how the distribution is defined. One approach uses a spliced (or piece-wise defined) severity distribution, with losses below a high threshold modeled by empirical distributions, and fitting of a parametric severity distribution only performed for losses above this top threshold [AK06]. Concretely, let \( F_b(x) \) be the severity cumulative distribution function (CDF) for the body of the distribution (e.g. an empirical distribution), and let \( F_t(x) \) be the severity CDF for the tail distribution (e.g. a parametric heavy-tailed distribution). Let \( T \) be the splicing threshold where the body and tail are joined. Then the spliced distribution \( F(x) \) is defined as

\[
F(x) = \begin{cases} 
F_b(x) & \text{if } x \leq T, \\
F_b(T) + (1 - F_b(T)) F_t(x) & \text{if } x > T.
\end{cases}
\]

With the exception of Extreme-Value Theory based models, the literature on severity distributions used in OpVaR models often assumes a non-spliced severity distribution, and hence distributions are fitted to all available loss data (or data above a relatively low collection threshold); see e.g. [DP07, OC12, CMAU09]. The specific composition of a severity distribution (spliced or non-spliced) impacts both MLE fitting error (in the spliced case, we fit to a smaller number of larger losses compared to non-spliced) and how this error translates to OpVaR (Section 3).

2.2. **Asymptotic normality of MLE.** Informally, asymptotic normality of MLE says that the distribution of fitted parameters to data will be normally distributed, centered about the true parameters, with a prescribed covariance matrix that depends on the sample-size. Let \( X = (x_1, \ldots, x_n) \) be data from an underlying distribution with probability density function (PDF) \( f(x|\theta^*) \), where \( \theta^* \) are the true parameters. Then to test if asymptotic normality holds for this distribution, we
need to test if, as \( n \) increases, MLE yields fitted parameters \( \hat{\theta} \) that are normally distributed. This property can be tested with parametric bootstrapping, that is, for a fixed \( n \), we sample \( n \) data points from the true distribution \( f(x|\theta^*) \), and apply MLE to get \( \hat{\theta}_{1,n} \). We repeat this sample/fit procedure \( m \) times to obtain fitted parameters \( (\hat{\theta}_{1,n}, \ldots, \hat{\theta}_{m,n}) \), i.e. we have generated statistics for MLE fitted parameters for a sample size of \( n \).

Before giving a precise statement of asymptotic normality for MLE, we first chronicle the regularity assumptions required for its proof. Define the log-likelihood function of the distribution as \( \ell(\theta) = \log f(x|\theta) \). The notation \( E_{\theta^*} [g(x|\theta)] \) for a function \( g(x|\theta) \) means

\[
E_{\theta^*} [g(x|\theta)] = \int g(u|\theta^*) f(u|\theta^*) du.
\]

Then the usual regularity conditions are [CH79, Gre11]

1. The first three partial derivatives of \( \ell(x|\theta) \) with respect to \( \theta \) are continuous and finite for almost all \( x \) and for all \( \theta \) in a neighborhood of \( \theta^* \).
2. For all \( \theta_j \), and \( i = 1, 2, 3 \),

\[
E_{\theta^*} \left[ \frac{\partial \ell(x|\theta)}{\partial (\theta_j)^i} \right] < \infty.
\]

3. There exists a distribution function \( M(x) \) such that \( \| \frac{\partial \ell(x|\theta)}{\partial \theta_k} \| < M(x) \) for all \( \theta \) in a neighborhood of \( \theta^* \), and \( E_{\theta^*} [M(x)] < \infty \).

The Fisher information matrix for a \( k \)-parameter distribution is the \( k \times k \) matrix whose \((i,j)\) entry is

\[
(I(\theta^*))_{i,j} = E_{\theta^*} \left[ \left( \frac{\partial}{\partial \theta_i} \ell(x|\theta) \right) \left( \frac{\partial}{\partial \theta_j} \ell(x|\theta) \right) \right].
\]

Given the regularity conditions above, the Fisher information matrix admits the following simpler description:

\[
(I(\theta^*))_{i,j} = -E_{\theta^*} \left[ \frac{\partial^2}{\partial \theta_i \partial \theta_j} \ell(x|\theta) \right]
\]

A further requirement for asymptotic normality is identifiability (this requirement is usually invoked when discussing the consistency of MLE, which states that the estimated parameters converge to the true parameters in probability as the sample-size goes to infinity). Informally, a parametric distribution family is identifiable if the parameter \( \theta \) uniquely determines the distribution (i.e. no two different parameter values yield the same distribution). More precisely, a distribution family is identifiable if for any \( \theta_1 \neq \theta_2 \), there exists \( X = (x_1, \ldots, x_n) \) for some \( n \) such that \( f(X|\theta_1) \neq f(X|\theta_2) \).

Proving identifiability can be challenging, but if the moments of \( f(x|\theta) \) have a nice form, one strategy is as follows: Suppose for a contradiction that \( \theta \neq \theta' \), and \( X \sim f(x|\theta) \), \( X' \sim f(x|\theta') \). If further there exists a \( k \in \mathbb{N} \) such that \( E[X^k] \neq E[X'^k] \), then it follows that there exists a subset of the parameters space \( U \) of non-zero measure such that \( f(x|\theta) \neq f(x|\theta') \) for all \( x \in U \), and hence the distribution family is identifiable.

A further requirement is that the Fisher information matrix be non-singular in a neighborhood of \( \theta^* \). The study of when this condition fails has led to recent
interaction between statistical learning theory and geometry, since the Fisher information matrix can be interpreted as a metric on the parameter space. Work of Sumio Watanabe and others develops a theory reconstructing many of the desirable properties of MLE in the case of singular Fisher information matrices by resolution of singularities from algebraic geometry [Wat09, Wat13].

The final requirement for asymptotic normality to hold is that the model has to be correctly specified: if MLE is applied for one parametric distribution to fit data coming from a different distribution, then of course results appealing to the “true” parameters will be suspect.

Theorem 2.1 (Asymptotic normality of MLE). Under the conditions above, the MLE \( \hat{\theta} \) is asymptotically normal:

\[
\sqrt{n}(\hat{\theta} - \theta^*) \overset{d}{\to} N(0, I(\theta^*)^{-1}),
\]

where convergence is in distribution.

A proof of the theorem can be found in [Wal43]; sketch proofs are more abundant (see e.g. [CH79]).

Our main interest is in the asymptotic nature of this result. For an example of what can go wrong for finite sample-sizes, consider data \((x_1, \ldots, x_n)\) sampled independently from the normal distribution \(N(\mu, \sigma)\). Then MLE produces the estimator for the variance \(\hat{\sigma}^2 = 1/n \sum_{i} (x_i - \hat{\mu})^2\), which is biased for finite \(n\).

This theorem gives a natural interpretation of the Fisher information matrix, which informally encodes how much information about the distribution is contained in each of the parameter directions. For simplicity, assume that the Fisher information is diagonal. Then large entries in the Fisher information matrix (high levels of information) correspond in Theorem 2.1 to small variations for MLE parameter estimation. In fact, a standard method to estimate MLE variance in numerical solvers is to calculate the Fisher information matrix at the optimal parameters, and invert it as in Theorem 2.1. As a corollary, such variance estimates are in general only valid insofar as Theorem 2.1 applies, in particular, under the assumption of large sample-sizes.

The higher-order regularity conditions above can be challenging to check in practice, and lack an obvious statistical interpretation. It can happen that the conditions of Theorem 2.1 are not satisfied, yet asymptotic normality still holds [LC70, Smi85]. Moreover, if the Fisher information matrix is singular (and not identically of determinant 0), then the set of parameters for which it is singular is of co-dimension at least one in the space of parameters (this is the solution set of \(\text{det} I(\theta) = 0\)). Hence for almost all parameters \(\theta\), the Fisher information matrix will be non-singular. Particular care is thus warranted when applying MLE to the generalized Beta distribution of the second kind, described in the next section, since the assumptions about the Fisher information matrix are difficult to verify.

A further challenge in bridging theory and practice is that for all but a few distributions, the algorithms used to determine the optimal parameters are numerical, and may produce only a local maximum of the log-likelihood function. As we describe the severity distributions under consideration below, we will thus also sketch the algorithms used in MLE and their potential shortcomings.

2.3. Heavy-tailed distributions. The severity distributions used in OpVaR models are generally heavy-tailed. We will take heavy-tailed to mean subexponential
(definition below), but several other definitions exist in the literature. For the convenience of the reader, we also sketch other common definitions and, where possible, relate them to one another.

**Definition 2.2.** Let $F$ be a cumulative distribution function with support in $(0, \infty)$. Then $F$ is subexponential if, for all $n \geq 2$,

$$\lim_{x \to \infty} \frac{F_n^*(x)}{F(x)} = n,$$

where $F(x) = 1 - F(x)$ is the tail, or survival function, and the numerator in the limit is the $n$-fold convolution of $F(x)$.

All of the distributions considered here are subexponential (for the Weibull distribution, this holds when the shape parameter is less than one; see Section 2.3.2 below).

Subexponentiality implies another property that is sometimes taken as the definition of heavy-tailed, namely that the tail decays more slowly than any exponential function. With the notation as above, the precise formulation is that for all $\epsilon > 0$,

$$\lim_{x \to \infty} e^{\epsilon x} \bar{F}(x) = \infty.$$

See [EKM97], Lemma 1.3.5 (b) for the proof that a subexponential distribution function satisfies the above limit.

An important subclass of subexponential distributions consists of regularly varying functions:

**Definition 2.3.** A positive, Lebesgue measurable function $f$ on $(0, \infty)$ is regularly varying at $\infty$ with index $\alpha \in \mathbb{R}$ if

$$\lim_{x \to \infty} \frac{f(tx)}{f(x)} = t^\alpha$$

for all $t > 0$.

Distributions with regularly varying tails exhibit one of the general properties expected of heavy-tailed distributions on the level of aggregate losses, namely that the tail of the maximum determines the tail of the sum. Note that the lognormal and Weibull distributions are not regularly varying.

For a regularly varying $F$ with tail index $\alpha > 0$, all moments of the associated random variable higher than $\alpha$ will be unbounded; see [EKM97] Proposition A.3.8 (d). Hence regular variation implies one final characterization of heavy tails. Some sources differentiate between heavy and “light”-tailed distributions based on the existence of finite moments [DFRJ07]. Under this classification, the lognormal and Weibull distributions are light-tailed since all moments are finite, while the Pareto, log-logistic and GB2 distributions all have infinite moments, and are considered in this usage to be heavy-tailed.

One subclass of severity distributions we do not consider below arises from Extreme Value Theory, including the Generalized Extreme Value (GEV) and Generalized Pareto distributions. These distributions are generally not calibrated via MLE, but rather with methods from Extreme Value Theory, such as Peaks-Over-Threshold (see [EKM97]). We thus limit our distributions to heavy-tailed distributions for which MLE is a prominent fitting method. Besides Extreme Value Theory distributions, we also pass over the $g$-and-$h$ distribution, despite recent attention
in the literature, since there is no closed-form for its PDF, and is most naturally fitted to data by quantile-matching [DP07] (MLE fitting methods do exist, however [RM02]).

The supports of the distribution families considered here may vary, which poses a problem when fitting loss data, especially in the case of a spliced distributions we study. For the Pareto distribution, one of the parameters defines the support, which contradicts an assumption required for the proof of asymptotic normality of MLE. Hence we set the parameter to the splice location, $T$, thus making the Pareto distribution a one-parameter family.

For the lognormal, log-logistic and Generalized Beta distribution of the second kind, the support is the positive real numbers. For fitting a spliced distribution, there are two standard approaches: replace the distribution with either the shifted or the truncated version to ensure that the support is contained in the tail region of the spliced distribution. Truncated distributions pose considerable problems for MLE fitting [OC12], so we consider exclusively shifted versions of these distributions.

2.3.1. Pareto Distribution. As mentioned above, the Pareto distribution is typically defined as a 2-parameter family, but since its support depends on one parameter (typically called the scale parameter, $T$), we fix this as the threshold of our spliced severity distribution (e.g. $T = 100000$), and consider the Pareto distribution as depending on one parameter, the scale, $\alpha$, resulting in PDF

$$f(x|\alpha) = \frac{\alpha T^\alpha}{x^{\alpha+1}},$$

where $x \geq T$, and is 0 otherwise. For $X \sim \text{Pareto}(\alpha)$, note that the first moment of $X$ is bounded if and only if $\alpha > 1$, and the variance is bounded only for $\alpha > 2$.

It is easy to show that the Pareto distribution is identifiable for all values of $\alpha$, and the Fisher information matrix is the scalar $I(\theta) = I(\alpha) = 1/\alpha^2$, which is indeed a positive-definite matrix (of size $1 \times 1$). The unique solution to the likelihood equation $\nabla \ell = 0$ is

$$\hat{\alpha} = \frac{n}{\sum^n \log(x_i/T)},$$

hence no numerical solver is required to perform MLE for the Pareto distribution.

2.3.2. Weibull distribution. The Weibull distribution is a generalization of the exponential distribution. For shape and scale parameters $a, b > 0$, the PDF is

$$f(x|a,b) = (a/b) (x/b)^{a-1} \exp \left(- \left(\frac{x}{b}\right)^a\right)$$

In [Wei07], the three-parameter Weibull distribution is considered, with an extra location parameter $u$ that determines the support. As noted above with reference to the Pareto distribution, a key assumption of MLE is that the support is independent of the parameters to be estimated. We thus consider the shifted distribution for Weibull, which is equivalent to setting the location parameter to $u = T$.

The Fisher information matrix for the Weibull distribution is

$$(2.3) \quad I(\theta) = I(a,b) = \begin{pmatrix} \frac{1}{a^2} \left( \psi'(1) + \psi^2(2) \right) & -\frac{1}{b} \left( 1 + \psi(1) \right) \\ -\frac{1}{b} \left( 1 + \psi(1) \right) & \frac{1}{a^2} \end{pmatrix};$$

see [CK06], but note the parametrization: some sources define the scale parameter as the reciprocal of what is given here.
The MLE properties of the Weibull distribution have been studied in [Smi85, Woo72, Aka75], although these works consider the three-parameter Weibull distribution for which MLE is especially problematic. It is shown that MLE is not even consistent if \( a \leq 1 \). For \( 1 < a < 2 \), MLE is not asymptotically normal, while if \( a = 2 \), MLE is asymptotically normal, but with different covariance matrix than that of Theorem 2.1. If \( a > 2 \), asymptotic normality holds (as well as asymptotic efficiency). Note that the Weibull distribution is heavy-tailed (i.e. subexponential) if and only if \( a < 1 \) ([EKM97], Example 1.4.3 for the if statement, while reverse implication follows from the existence of a Cramér-Lundberg exponent when \( a \geq 1 \)), hence MLE for subexponential Weibull distributions results in an inconsistent estimator.

The likelihood equations for the Weibull distribution can be solved explicitly. As will be seen in Section 2.4 (and [Lar15]), for all four loss data sets to which we apply MLE, the “true” values of \( \alpha \) are all less than one, leading to non-bounded Fisher information matrices. This manifests in the algorithm of fitdistrplus via warning messages that the resulting system is singular. In case of non-convergence of MLE, we discard the parameter estimate.

2.3.3. Lognormal Distribution. The lognormal distribution \( \log N(\mu, \sigma) \) has PDF

\[
f(x|\mu, \sigma) = \frac{1}{\sqrt{2\pi \sigma x}} \exp \left( -\frac{(\log x - \mu)^2}{2\sigma^2} \right)
\]

Note that we choose the second parameter to be \( \sigma \) and not \( \sigma^2 \). This convention makes a difference in calculating the Fisher information matrix, which is

\[
I(\theta) = I(\mu, \sigma) = \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 2/\sigma^2 \end{pmatrix}.
\]

The Fisher information matrix of the lognormal distribution is non-singular, since a random variable \( X \) is lognormal if and only if there is a normally distributed random variable \( Y \) with \( X = \exp(Y) \), and the function \( \exp : \mathbb{R} \to (0, \infty) \) is a diffeomorphism (hence, loosely speaking, all properties involving derivatives and integrals that hold for normally distributed variables hold for lognormal, and vice-versa).

The lognormal distribution is also identifiable for all allowed \( (\mu, \sigma) \), since the same holds for the normal distribution. The Fisher information matrix is positive-definite, since it is a diagonal matrix with positive entries. As with the Pareto distribution, the likelihood equations for the lognormal distribution can be solved explicitly, hence the determination of \( (\hat{\mu}, \hat{\sigma}) \) is computationally unproblematic.

2.3.4. Log-logistic Distribution. As with the lognormal distribution, a random variable \( X \) follows a log-logistic distribution if and only if there exists a logistic random variable \( Y \) such that \( X = \exp(Y) \). The log-logistic distribution \( LL(a, s) \) has PDF

\[
f(x|a, s) = \frac{a \left( \frac{x}{s} \right)^a}{x(1 + (x/s)^a)^2}
\]

for \( x \geq 0 \), and is zero otherwise. The parameters \( a \) and \( s \) must both be positive. Like the Pareto distribution, a log-logistic distributions can have an unbounded first moment, namely, when \( a \leq 1 \), while for \( a \leq 2 \), the variance is also unbounded.
The Fisher information matrix of \( X \sim LL(a, s) \) is (SMIT88)

\[
I(\theta) = I(a, s) = \begin{pmatrix} \frac{3 + \pi^2}{9a^2} & 0 \\ 0 & \frac{1}{3} (\frac{a}{s})^2 \end{pmatrix},
\]

which is positive-definite, since \( a, s > 0 \). Moreover, the log-logistic distribution is identifiable for \( a > 1 \), which follows from the above general strategy, since its median is \( s \) and its mode is \( s \left( \frac{a - 1}{a + 1} \right)^{1/a} \), which is strictly increasing in \( a \), and is hence injective.

To fit the log-logistic distribution, we use \texttt{fitdistrplus} as described for the Weibull distribution. For initial parameter values, we take (see e.g. J¨oh12)

\[
s_{\text{init}} = \text{Median}(x_1, \ldots, x_n) \\
a_{\text{init}} = \log(n - 1) / \log(\max(x_1, \ldots, x_n)/s_{\text{init}})
\]

2.3.5. Generalized Beta Distribution of the Second Kind. The GB2 distribution (also known as the transformed Beta distribution) is nested within the more general Feller-Pareto distribution \( FP(\mu, \sigma, \gamma, \gamma_1, \gamma_2) \), and itself nests the Weibull, lognormal, and log-logistic distributions (as well as the generalized Pareto and inverse Burr distributions [Bra02]), hence the GB2 distribution makes possible an evaluation of the trade-off between generality (GB2) and parsimony (Weibull, lognormal and log-logistic) when modeling OR loss data.

The GB2 distribution has PDF

\[
f(x) = \frac{a(x/b)^{ap-1}}{bB(p,q)(1 + (x/b)^a)^{p+q}},
\]

where \( B(p, q) \) is the Beta function (or Euler integral), defined for \( p, q > 0 \) as

\[
B(p, q) = \int_0^1 t^{p-1}(1 - t)^{q-1}dt,
\]

The \( m \)th moment of \( X \sim GB2(a, b, p, q) \) is \( \frac{b^m B(p + h/a, q - h/a)}{B(p, q)} \), and no moments are finite above the \( aq \)th one [BM87].

The Fisher information matrix for the GB2 distribution can be derived from that of the Feller-Pareto distribution [Bra02]. Specifically, since GB2(a, b, p, q) = FP(0, b, 1/a, q, p), we use the change-of-variable formula \( JI_{FP,J^T} \), where \( J \) is the Jacobian matrix of the coordinate change \( \mu \to 0, \sigma \to b, \gamma \to 1/a, \gamma_1 \to q, \) and \( \gamma_2 \to p \). Writing \( I = (I_{i,j}) = (I_{j,i}) \), the Fisher information matrix for the
GB2($a, b, p, q$) distribution has entries

$$
I_{1,1} = a^2 + \frac{a^2pq}{p + q + 1},
$$

$$
I_{1,2} = -\frac{pq(\psi(p) - \psi(q)) + q - p}{b(p + q - 1)},
$$

$$
I_{1,3} = \frac{q(\psi(p) - \psi(q)) - 1}{a(p + q)},
$$

$$
I_{1,4} = \frac{p(\psi(q) - \psi(p)) - 1}{a(p + q)},
$$

$$
I_{2,2} = \frac{a^2pq}{b^2(p + q + 1)},
$$

$$
I_{2,3} = \frac{aq}{b(p + q)},
$$

$$
I_{2,4} = -\frac{ap}{b(p + q)},
$$

$$
I_{3,3} = \psi'(p) - \psi'(p + q),
$$

$$
I_{3,4} = -\psi'(p + q),
$$

$$
I_{4,4} = \psi'(q) - \psi'(p + q),
$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function, and its derivative $\psi'(x)$ is the trigamma function.

We implement our own MLE for the GB2 distribution as follows. We incorporate the linear bounds on $p, q$ by implicit penalty in the likelihood function, and minimize the log likelihood function with the package NelderMead [BB14]. To obtain initial parameter values, we use the pseudo-MLE functionality of the package GB2 [GN14]. Since Nelder-Mead in general only returns a local minimum, we run the minimization at two other parameter start values, obtained by perturbing the loss data (losses shifted up for one, and shifted down for the other) and applying pseudo-MLE to the perturbed loss data. If at least one of the three start parameters leads to a convergent solution, we take the calibrated parameters corresponding to the lowest negative log-likelihood value.

Before turning to results, note that Theorem 2.1 assumes that we are able to find the global maximum of the likelihood function. For both the log-logistic and GB2 distributions, there is no guarantee of having found a global maximum. We hope that the descriptions of our methodology above will suffice to enable practitioners using the log-logistic or GB2 distributions to judge how their optimization algorithms differ.

2.4. Asymptotic normality for heavy-tailed distributions. In this section, we give an example of how asymptotic normality holds for the severity distributions described above when fitted to moderately heavy loss data, and examine what implications this has for approximating parameter confidence intervals with asymptotic normality. The corresponding results for three other sets of loss data can be found in [Lar15].

We now describe our parametric bootstrapping procedure in detail for a fixed loss data set $\text{losses}$ and sample size $n$.

For $\text{distn in \{pareto, Weibull, lognormal, log-logistic, GB2\}}$ with CDF $F(x|\theta)$
(1) Fit distn to losses with MLE to obtain true parameters $\theta^*$
(2) For $i \in \{1, \ldots, m\}$ (we take $m = 5000$)
   (a) Draw $n$ samples from the true distribution $F(x|\theta^*)$ to obtain bootstrap losses $\text{losses}_i$.
   (b) Fit distn to $\text{losses}_i$ with MLE to obtain bootstrapped parameters $\hat{\theta}_{i,n}$

For each distribution family and each sample-size $n$ we thus obtain statistics for parameter estimation. We then compare each component of the bootstrapped parameters $\hat{\theta}_{1,n}, \ldots, \hat{\theta}_{m,n}$ to the prediction of Theorem 2.1. For example, each $\hat{\theta}_{i,n}$ for a lognormal distribution will be a vector with two components, $\hat{\theta}_{i,n} = (\mu_{i,n}, \sigma_{i,n})$, and for each of these components we plot the corresponding kernel density estimate (essentially a smart histogram; see e.g. [HTF09], Chapter 6) against the normal distribution of Theorem 2.1 for this component. Continuing with this example, the normal distribution corresponding to the bootstrapped $\mu_{i,n}$ will have variance $1/(\sigma^*)^2$. (For readability, in the plots below we center the predicted normal distribution at the true value rather than 0, and move the factor of $\sqrt{n}$ to the right-hand side of the limit). Note that a comprehensive study of applicability of Theorem 2.1 would require more than the plots and confidence intervals we present, which only focus on the marginal distributions of the estimated parameters.

For the below example, the underlying loss data set consists of 2590 losses, 0.191119691119691 of which are above the splicing threshold of 100,000 EUR. The data are not particularly heavy, with mean of 131560, median of 39018, and no losses larger than 30m EUR. Results for heavier loss data are found in [Lar15].

2.4.1. Asymptotic normality plots. The plots for this loss-data set show that the bootstrapped parameters behave as in Theorem 2.1 even for sample-size as small as 75 for the Pareto and lognormal distributions (although the small-sample bias of the MLE estimator for $\sigma$ of the lognormal distribution can be seen). The Weibull, log-logistic and GB2 distributions show varying degrees of skewness. The skewness for the Weibull distribution does not decrease with increasing sample-size, as it does for the others. This behavior is not surprising, however, since the true shape parameter is 0.56, i.e. the MLE is not consistent for this value, let alone asymptotically normal.

![Figure 1. UOM1: AN for Pareto: true $\theta = (1.11)$, sample-size 75, and 5000 bootstraps](image-url)
Figure 2. UOM1: AN for Weibull: true $\theta = (0.56, 212303.18)$, sample-size 5000, and 5000 bootstraps

Figure 3. UOM1: AN for lognormal: true $\theta = (11.3, 1.8)$, sample-size 75, and 5000 bootstraps

Figure 4. UOM1: AN for log-logistic: true $\theta = (1, 84000)$, sample-size 75, and 5000 bootstraps
Figure 5. UOM1: AN for GB2: true $\theta = (0.837, 117516.887, 1.184, 1.454)$, sample-size 75, and 5000 bootstraps

Figure 6. UOM1: AN for GB2: true $\theta = (0.837, 117516.887, 1.184, 1.454)$, sample-size 5000, and 5000 bootstraps
2.4.2. **Approximating parameter confidence intervals with asymptotic normality.**

We now turn to question [1] from the introduction about the use of Theorem 2.1 to approximate parameter confidence intervals. The below table shows the percent error of using this approximation relative to the “true” 95% confidence intervals obtained by quantiles of the bootstrapped parameters, i.e. we compare the difference of the 97.5% quantile and 2.5% quantile from 5000 bootstrapped parameters to same difference of quantiles from the normal distribution dictated by Theorem 2.1.

The results mirror what can be seen from the plots: for the Pareto, lognormal and log-logistic distributions, the normally approximated 95% confidence intervals are within a few percent of the true ones, while the approximation is relatively poor for the Weibull and GB2 distributions. For GB2, the normally approximated confidence intervals are within 10% of the true ones given enough data ($\geq 5000$ losses). The approximation for the Weibull distribution gets worse as sample sizes increase, a phenomenon that can also be seen from the plots.

**Table 1.** Percent error of 95% confidence intervals derived from asymptotic normality by sample-size

<table>
<thead>
<tr>
<th>Distribution</th>
<th>75</th>
<th>200</th>
<th>500</th>
<th>1500</th>
<th>2500</th>
<th>5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pareto shape</td>
<td>1.5%</td>
<td>3.1%</td>
<td>0.45%</td>
<td>-1%</td>
<td>-0.9%</td>
<td>-2.2%</td>
</tr>
<tr>
<td>Weibull shape</td>
<td>8.2%</td>
<td>5.6%</td>
<td>4.8%</td>
<td>4.2%</td>
<td>4.8%</td>
<td>8%</td>
</tr>
<tr>
<td>Weibull scale</td>
<td>7.4%</td>
<td>8.6%</td>
<td>10.4%</td>
<td>17.5%</td>
<td>21.4%</td>
<td>31%</td>
</tr>
<tr>
<td>lognormal meanlog</td>
<td>1.6%</td>
<td>0.94%</td>
<td>0.73%</td>
<td>1.38%</td>
<td>-0.573%</td>
<td>1.07%</td>
</tr>
<tr>
<td>lognormal sdlog</td>
<td>-1.8%</td>
<td>1.63%</td>
<td>1.07%</td>
<td>-0.43%</td>
<td>-0.075%</td>
<td>-0.64%</td>
</tr>
<tr>
<td>log-logistic shape</td>
<td>4.0%</td>
<td>0.67%</td>
<td>1.01%</td>
<td>0.59%</td>
<td>0.65%</td>
<td>0.81%</td>
</tr>
<tr>
<td>log-logistic scale</td>
<td>3.4%</td>
<td>-1.41%</td>
<td>-0.62%</td>
<td>-0.99%</td>
<td>2.02%</td>
<td>0.11%</td>
</tr>
<tr>
<td>GB2 shape1</td>
<td>71.70%</td>
<td>27%</td>
<td>8.8%</td>
<td>2.4%</td>
<td>3.7%</td>
<td>10%</td>
</tr>
<tr>
<td>GB2 scale</td>
<td>61.48%</td>
<td>68%</td>
<td>57.5%</td>
<td>20.1%</td>
<td>14.5%</td>
<td>7.9%</td>
</tr>
<tr>
<td>GB2 shape2</td>
<td>5.79%</td>
<td>37%</td>
<td>43.8%</td>
<td>20.8%</td>
<td>13.6%</td>
<td>9.0%</td>
</tr>
<tr>
<td>GB2 shape3</td>
<td>-0.91%</td>
<td>38%</td>
<td>48.9%</td>
<td>22.8%</td>
<td>15.4%</td>
<td>8.4%</td>
</tr>
</tbody>
</table>
3. OpVaR error analysis

Our ultimate interest is to estimate the effect of uncertainty in parameter estimation as a function of sample size on OpVaR, which in models considered here corresponds to a high quantile of the annual loss distribution (to be more precise, expected loss should be subtracted to obtain OpVaR). Given an annual loss distribution $G(x) = G(x, \theta)$, the goal is to understand statistical properties of $G^{-\alpha}(\alpha, \theta^*)$ for $\alpha$ close to 1 in relation to the number of losses used to fit $\theta^*$. As an application, we construct confidence bounds about the true OpVaR for the severity distributions under consideration and for varying sample-sizes.

The most direct method to construct confidence intervals would be to calculate the chosen high quantile (e.g. $\alpha = 0.999$) for each of the bootstrapped parameters from the previous section (e.g. via a Monte Carlo simulation). Instead, we will employ the single-loss approximation for $G^{-\alpha}(\alpha, \theta^*)$ developed in [BK05], thus avoiding the computational effort required to simulate loss distributions for each bootstrapped parameter.

We assume for simplicity that the loss-process has arrival times that can be modeled as Poisson random variables, and that the time-window over which we model losses is one year (both assumptions can be relaxed; see [BK05]). If $\lambda$ is the resulting intensity (e.g. the average number of losses per year, assuming no time-weighting of loss frequencies), then the main result of [BK05] hinges on the asymptotic relation $G(x) \sim \lambda \bar{F}(x)$ as $x \to \infty$, which holds for subexponential severity distributions.

**Theorem 3.1** (Theorem 2.4, [BK05]). Let $G(x)$ be the loss distribution of the standard Poisson LDA model with intensity $\lambda$ and subexponential severity distribution $F(x)$. Then the quantiles of the loss distribution satisfy

$$G^{-\alpha}(\alpha) = F^{-\alpha} \left(1 - \frac{1 - \alpha}{\lambda}(1 + o(1))\right) \text{ as } \alpha \to 1$$

**Definition 3.2.** For loss distribution $G$ as in Theorem 3.1, we define the single-loss approximation to the $\alpha$ quantile of $G$ as

$$G^{-\alpha}(\alpha) \approx F^{-\alpha} \left(1 - \frac{1 - \alpha}{\lambda}\right)$$

This result assumes that the severity distribution is fitted to all loss data. If, however, a spliced severity distribution is employed (e.g. empirical distribution below a threshold $T$, parametric tail above $T$), then the approximation must be modified.

**Corollary 3.3.** Let $F(x)$ be a spliced distribution, with (body) distribution $F_b(x)$ for $x < T$ and (tail) distribution $F_t(x)$ for $x \geq T$. If $p_T = F_b(T) < 1$, then the single-loss approximation for the loss distribution $G(x)$ is

$$G^{-\alpha}(\alpha) \approx F^{-\alpha}_t \left(1 - \frac{1 - \alpha}{(1 - p_T)\lambda}\right)$$

Note that the case of $p_T = 1$ is not interesting in the present context, since such a spliced severity distribution is not heavy-tailed, for in this case $F(x \geq T) = 0$.

For non-sliced distributions, one intuition behind the single-loss distribution is that only the tail (i.e. high quantiles) of the severity distribution matters for Op-VaR. For spliced distributions with a given threshold $T$, the above corollary says
that the only effect the body has is through the probability of a loss being above $T$; all other properties of the body distribution play no role in the approximation.

The proof of this corollary is a straightforward calculation, which we give below for the convenience of the reader (see also [Deg10], Section 4). In case the body severity distribution $F_b(x)$ is an empirical CDF and all losses are given the same weight, then the difference between the usual single-loss approximation and that of Corollary 3.3 has a simple interpretation: for the spliced distribution, $(1 - p_T)$ is the probability that a loss will occur in the tail, so the intensity of interest is no longer $\lambda$, but rather $(1 - p_T)\lambda$, which is the expected number of losses in the tail.

**Proof.** If $p_T = 1$, the spliced distribution for $x \geq T$ has CDF

$$F(x) = p_T + F_t(x)(1 - p_T),$$

and so $F_t(x) = \frac{F(x) - p_T}{1 - p_T}$. Hence solving $\alpha = F(q)$ for $q$ gives

$$q = F^{\leftarrow}(\alpha) = F^{\leftarrow}_t\left(\frac{\alpha - p_t}{1 - p_T}\right).$$

The result now follows by substituting into the usual single-loss approximation. □

If the tail distribution is shifted, e.g., for $x \geq T$, the spliced severity distribution is $F(x) = p_T + F_t(x - T)(1 - p_T)$, then the single-loss approximation becomes

$$G^{\leftarrow}(\alpha) \approx F^{\leftarrow}_t\left(1 - \frac{1 - \alpha}{(1 - p_T)\lambda}\right) + T.$$

There are further refinements of the single-loss approximation in the literature [BS06, Deg10] that incorporate a correction terms involving the expectation value of the severity distribution. The documented shortcomings of the original single-loss approximation apply mostly to lighter severity distributions and high frequencies. Since we consider here spliced distributions with a tail threshold of 100k EUR, our effective frequency is relatively low, and our fitted distributions are heavier than for non-spliced distributions, hence we do not want to exclude a priori severity distributions with infinite expectation value. For these reasons, along with our emphasis on relative stability of OpVaR (more on this below), we employ the original single-loss approximation. We nevertheless compare the single-loss approximation to Monte Carlo simulation in the results of the next section.

### 3.1. OpVaR confidence intervals.

We now apply the spliced-version of the single-loss-approximation to the bootstrapped parameters of Section 2.4 to obtain OpVaR statistics. We choose the splice threshold to be $T = 100k$ EUR, and take $\alpha = 0.999$, which corresponds to regulatory capital in Pillar 1 of the Basel 2 regulations.

For the present, relatively light loss data, as well as the other three loss data sets in [Lar15], we give the (possibly infinite) mean of the true severity distribution, and the true OpVaR($\alpha$) using both the approximation of Corollary 3.3 and a Monte Carlo simulation (with 1mil MC steps). As reported in [BS06, Deg10], the quality of the single-loss approximation gets worse as the frequency increases; see e.g. Figure 1 of [Deg10]. In the present context of a spliced distribution, our frequencies are relatively low, since relevant frequency used in Corollary 3.3 corresponds to the frequency of high losses (i.e. above the threshold of 100,000). Secondly, although not explicitly stated, the quality of the approximation seems to improve the heavier the severity distribution is. This trend is most clearly seen in [BS06] for Pareto severity distributions in Figure 4. To be more precise, the
adjustments of [BS06, Deg10] add a positive correction term involving the mean of the severity distribution. Assuming a finite mean, the ratio of high quantiles to the mean will typically be higher for heavier distributions, and hence we would expect the correction term to be less significant for heavier distributions than lighter ones.

With the exception of the (relatively light) Weibull distribution, the spliced single-loss approximation performs quite well as compared to the Monte Carlo OpVaR(α). The single-loss approximation for the Weibull distribution results in underestimation of OpVaR(α) by approximately 30%. Since we focus on relative difference between true OpVaR(α) and bootstrapped OpVaR(α), the single-loss approximation may still give a meaningful picture of stability. To be on the safe side, however, we report simulation results for the Weibull distribution, but due to the questionable performance of the single-loss approximation, all conclusions about suitability or lack thereof for the Weibull distribution for small sample-sizes are flagged with an asterisk. It would be interesting to explore the adjusted single-loss approximations of [BS06, Deg10].

We now describe the simulations and analyses in more detail. For each distribution and sample-size, we take the bootstrapped parameters θ₁,...,θₙ (we take N = 5000) and calculate the spliced single-loss approximation to yield bootstrapped OpVaR statistics, OpVaR(α)₁,...,OpVaR(α)ₙ. The frequency contribution to the single-loss approximation (essentially, the number of losses in the tail) derives from the underlying data, and hence stays constant across the different combinations of severity and sample-size. There may be objections to holding the frequency fixed yet varying the sample-size for a single UOM. In OpVaR models such as in [AK06], however, frequencies are calibrated independent of loss amounts and only on internal loss data. On the severity side, the regulatory requirement to incorporate external data, coupled with other modeling decisions such as splicing thresholds and UOM clustering, justify the interest in varying sample-sizes while holding frequencies fixed.

For each sample-size, we calculate two types of summary statistics. In the first, we give absolute mean, media and standard deviations, as well as relative versions. For example, the relative “mean bias” of OpVaR(α)₁,...,OpVaR(α)ₙ is

\[
\text{avg}(\text{OpVaR}(\alpha)_i) - \text{OpVaR}(\alpha)_{true} \big/ \text{OpVaR}(\alpha)_{true} \%.
\]

This mean bias is a central object of study in [OC12], where they claim that MLE results in capital overestimation. The meaning of this statistic for modeling decisions, however, is not completely clear. This point will be discussed in further detail in Section 3 but to pave the way, we also consider the median of the OpVaR(α)i. It is interesting to note that, with the exception of the GB2 distribution, the mean bias is positive, while the median bias is negative (for GB2, both types of bias are positive).

The second set of statistics given for each distribution consists of relative confidence bounds as a function of sample-size. The particular quantiles of the bootstrapped OpVaR(α)i values were chosen to match the usual confidence intervals of 90%, 80% and 68% (i.e. the rule-of-thumb one standard-deviation interval). An immediate observation is that these confidence bounds are not symmetric, and hence any claims about model stability based purely on the standard deviation (or another symmetric statistic) would obscure the differing impacts of underestimating vs overestimating OpVaR.
Next, we give the mean and OpVaR for each of the distributions fitted to the loss data, where OpVaR is calculated both via the single-loss approximation and by Monte Carlo simulation.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Mean</th>
<th>VaR(0.999), SLA</th>
<th>VaR(0.999), MC sim</th>
</tr>
</thead>
<tbody>
<tr>
<td>pareto</td>
<td>1.01</td>
<td>212.99</td>
<td>217.13</td>
</tr>
<tr>
<td>weibull</td>
<td>0.10</td>
<td>9.80</td>
<td>14.02</td>
</tr>
<tr>
<td>lnorm</td>
<td>0.30</td>
<td>47.32</td>
<td>51.13</td>
</tr>
<tr>
<td>llogis</td>
<td>66.37</td>
<td>411.36</td>
<td>415.41</td>
</tr>
<tr>
<td>gb2</td>
<td>0.10</td>
<td>152.32</td>
<td>156.35</td>
</tr>
</tbody>
</table>

We now look at the two sets of OpVaR statistics described above for each severity distribution. The results will be discussed in fuller detail in Section 4, but we make a few preliminary observations here. For all distributions (both below and for the UOMs in the Appendix [Lar15]), the mean bias of OpVaR is positive, approaching zero as the sample size grows, while the median bias is negative. One explanation for the positive mean bias is given in [OC12], and will be discussed in Section 4.

Regarding the relative VaR confidence bounds, the first thing to note is the asymmetry. The relative overestimation error is in general larger than the underestimation error, a phenomenon that accords well with the above-mentioned mean bias. This asymmetry, which is especially strong for small sample-sizes, highlights the limited value of symmetric statistics (such as standard-deviation) in studying OpVaR models. In Section 4, we therefore focus on stability criteria based on relative error confidence bounds.

Our goal of comparing stability of different severity distributions is complicated by the sometimes very different OpVaR estimates of the distributions, even when calibrated to the same underlying data. We partially address this issue by focusing on relative OpVaR. A second, related, complication is our working assumption that each of the severity distributions has already passed some goodness-of-fit tests. For the present loss-data set, QQ-plots reveal that the MLE fitted Weibull distribution is too light compared to the highest losses, while the lognormal has a good fit across loss amounts, and the remaining loss distributions tend to overestimate the highest losses. The fit of the log-logistic distribution for the smaller losses is quite poor. For a severity distribution that poorly fits the data, it is to be expected that the stability of the resulting OpVaR estimates will also be negatively effected. The results given below should therefore be taken as a framework, and not a recipe book. By studying relative OpVaR and looking at different loss data sets (see [Lar15]), we aim to show how stability can be accurately quantified and used as a criterion for model development, but the practitioner will need to perform these analyses her or himself, especially with an eye to goodness-of-fit.
### Table 3. Pareto VaR summary statistics (in millions) for 5000 bootstraps, true parameters $\theta = (1.11)$, and true VaR = 213m

<table>
<thead>
<tr>
<th></th>
<th>mean VaR</th>
<th>mean bias</th>
<th>median VaR</th>
<th>median bias</th>
<th>sd VaR</th>
<th>relative sd</th>
</tr>
</thead>
<tbody>
<tr>
<td>75 samples</td>
<td>319</td>
<td>49.8%</td>
<td>202</td>
<td>-4.95%</td>
<td>395</td>
<td>185%</td>
</tr>
<tr>
<td>500 samples</td>
<td>225</td>
<td>5.64%</td>
<td>212</td>
<td>-0.565%</td>
<td>80.1</td>
<td>37.6%</td>
</tr>
<tr>
<td>1000 samples</td>
<td>219</td>
<td>2.65%</td>
<td>212</td>
<td>-0.442%</td>
<td>52.9</td>
<td>24.8%</td>
</tr>
<tr>
<td>3000 samples</td>
<td>215</td>
<td>0.815%</td>
<td>212</td>
<td>-0.362%</td>
<td>30.3</td>
<td>14.2%</td>
</tr>
<tr>
<td>5000 samples</td>
<td>214</td>
<td>0.632%</td>
<td>213</td>
<td>0.0686%</td>
<td>23.1</td>
<td>10.8%</td>
</tr>
</tbody>
</table>

### Table 4. Pareto VaR confidence bounds (relative difference) for 5000 bootstraps, true parameters $\theta = (1.11)$, and true VaR = 213m

<table>
<thead>
<tr>
<th></th>
<th>5%</th>
<th>10%</th>
<th>26%</th>
<th>84%</th>
<th>90%</th>
<th>95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>75 samples</td>
<td>-75%</td>
<td>-67%</td>
<td>-44%</td>
<td>138%</td>
<td>218%</td>
<td>340%</td>
</tr>
<tr>
<td>500 samples</td>
<td>-43%</td>
<td>-36%</td>
<td>-21%</td>
<td>40%</td>
<td>55%</td>
<td>77%</td>
</tr>
<tr>
<td>1000 samples</td>
<td>-33%</td>
<td>-26%</td>
<td>-14%</td>
<td>26%</td>
<td>35%</td>
<td>49%</td>
</tr>
<tr>
<td>3000 samples</td>
<td>-20%</td>
<td>-16%</td>
<td>-9%</td>
<td>15%</td>
<td>20%</td>
<td>27%</td>
</tr>
<tr>
<td>5000 samples</td>
<td>-16%</td>
<td>-13%</td>
<td>-7%</td>
<td>11%</td>
<td>15%</td>
<td>20%</td>
</tr>
</tbody>
</table>

### Table 5. Weibull VaR summary statistics (in millions) for 5000 bootstraps, true parameters $\theta = (0.56, 212303.18)$, and true VaR = 9.8m

<table>
<thead>
<tr>
<th></th>
<th>mean VaR</th>
<th>mean bias</th>
<th>median VaR</th>
<th>median bias</th>
<th>sd VaR</th>
<th>relative sd</th>
</tr>
</thead>
<tbody>
<tr>
<td>75 samples</td>
<td>10</td>
<td>2.06%</td>
<td>9.36</td>
<td>-4.5%</td>
<td>3.6</td>
<td>36.7%</td>
</tr>
<tr>
<td>500 samples</td>
<td>9.91</td>
<td>1.07%</td>
<td>9.81</td>
<td>0.0896%</td>
<td>1.31</td>
<td>13.4%</td>
</tr>
<tr>
<td>1000 samples</td>
<td>9.87</td>
<td>0.646%</td>
<td>9.8</td>
<td>0.002%</td>
<td>0.926</td>
<td>9.44%</td>
</tr>
<tr>
<td>3000 samples</td>
<td>9.87</td>
<td>0.731%</td>
<td>9.85</td>
<td>0.51%</td>
<td>0.559</td>
<td>5.7%</td>
</tr>
<tr>
<td>5000 samples</td>
<td>9.87</td>
<td>0.674%</td>
<td>9.85</td>
<td>0.512%</td>
<td>0.444</td>
<td>4.53%</td>
</tr>
</tbody>
</table>

### Table 6. Weibull VaR confidence bounds (relative difference) for 5000 bootstraps, true parameters $\theta = (0.56, 212303.18)$, and true VaR = 9.8m

<table>
<thead>
<tr>
<th></th>
<th>5%</th>
<th>10%</th>
<th>26%</th>
<th>84%</th>
<th>90%</th>
<th>95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>75 samples</td>
<td>-45%</td>
<td>-38%</td>
<td>-23%</td>
<td>36%</td>
<td>49%</td>
<td>70%</td>
</tr>
<tr>
<td>500 samples</td>
<td>-19%</td>
<td>-15%</td>
<td>-8%</td>
<td>14%</td>
<td>19%</td>
<td>25%</td>
</tr>
<tr>
<td>1000 samples</td>
<td>-14%</td>
<td>-11%</td>
<td>-6%</td>
<td>10%</td>
<td>13%</td>
<td>17%</td>
</tr>
<tr>
<td>3000 samples</td>
<td>-8%</td>
<td>-6%</td>
<td>-3%</td>
<td>6%</td>
<td>8%</td>
<td>10%</td>
</tr>
<tr>
<td>5000 samples</td>
<td>-6%</td>
<td>-5%</td>
<td>-2%</td>
<td>5%</td>
<td>7%</td>
<td>9%</td>
</tr>
</tbody>
</table>
Table 7. Lognormal VaR summary statistics (in millions) for 5000 bootstraps, true parameters $\theta = (11.3, 1.8)$, and true VaR = 47.3m

<table>
<thead>
<tr>
<th>Samples</th>
<th>Mean VaR</th>
<th>Mean Bias</th>
<th>Median VaR</th>
<th>Median Bias</th>
<th>Std VaR</th>
<th>Relative SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>75</td>
<td>51.9</td>
<td>9.64%</td>
<td>44.2</td>
<td>-6.64%</td>
<td>31.5</td>
<td>66.7%</td>
</tr>
<tr>
<td>500</td>
<td>48</td>
<td>1.48%</td>
<td>46.9</td>
<td>-0.899%</td>
<td>10.5</td>
<td>22.1%</td>
</tr>
<tr>
<td>1000</td>
<td>47.5</td>
<td>0.373%</td>
<td>46.9</td>
<td>-0.841%</td>
<td>7.33</td>
<td>15.5%</td>
</tr>
<tr>
<td>3000</td>
<td>47.5</td>
<td>0.408%</td>
<td>47.3</td>
<td>-0.00629%</td>
<td>4.14</td>
<td>8.74%</td>
</tr>
<tr>
<td>5000</td>
<td>47.4</td>
<td>0.27%</td>
<td>47.4</td>
<td>0.205%</td>
<td>3.24</td>
<td>6.84%</td>
</tr>
</tbody>
</table>

Table 8. Lognormal VaR confidence bounds (relative difference) for 5000 bootstraps, true parameters $\theta = (11.3, 1.8)$, and true VaR = 47.3m

<table>
<thead>
<tr>
<th>Samples</th>
<th>5%</th>
<th>10%</th>
<th>26%</th>
<th>84%</th>
<th>90%</th>
<th>95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>75</td>
<td>-62%</td>
<td>-53%</td>
<td>-35%</td>
<td>64%</td>
<td>93%</td>
<td>136%</td>
</tr>
<tr>
<td>500</td>
<td>-30%</td>
<td>-25%</td>
<td>-13%</td>
<td>23%</td>
<td>31%</td>
<td>42%</td>
</tr>
<tr>
<td>1000</td>
<td>-23%</td>
<td>-19%</td>
<td>-10%</td>
<td>16%</td>
<td>20%</td>
<td>28%</td>
</tr>
<tr>
<td>3000</td>
<td>-13%</td>
<td>-11%</td>
<td>-5%</td>
<td>9%</td>
<td>12%</td>
<td>16%</td>
</tr>
<tr>
<td>5000</td>
<td>-11%</td>
<td>-8%</td>
<td>-4%</td>
<td>7%</td>
<td>9%</td>
<td>12%</td>
</tr>
</tbody>
</table>

Table 9. Log-logistic VaR summary statistics (in millions) for 5000 bootstraps, true parameters $\theta = (1, 84000)$, and true VaR = 411m

<table>
<thead>
<tr>
<th>Samples</th>
<th>Mean VaR</th>
<th>Mean Bias</th>
<th>Median VaR</th>
<th>Median Bias</th>
<th>Std VaR</th>
<th>Relative SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>75</td>
<td>542</td>
<td>31.8%</td>
<td>371</td>
<td>-9.85%</td>
<td>588</td>
<td>143%</td>
</tr>
<tr>
<td>500</td>
<td>433</td>
<td>5.35%</td>
<td>407</td>
<td>-1.07%</td>
<td>150</td>
<td>36.4%</td>
</tr>
<tr>
<td>1000</td>
<td>419</td>
<td>1.93%</td>
<td>409</td>
<td>-0.67%</td>
<td>98.8</td>
<td>24%</td>
</tr>
<tr>
<td>3000</td>
<td>414</td>
<td>0.598%</td>
<td>410</td>
<td>-0.254%</td>
<td>55.5</td>
<td>13.5%</td>
</tr>
<tr>
<td>5000</td>
<td>415</td>
<td>0.862%</td>
<td>413</td>
<td>0.315%</td>
<td>42.9</td>
<td>10.4%</td>
</tr>
</tbody>
</table>

Table 10. Log-logistic VaR confidence bounds (relative difference) for 5000 bootstraps, true parameters $\theta = (1, 84000)$, and true VaR = 411m

<table>
<thead>
<tr>
<th>Samples</th>
<th>5%</th>
<th>10%</th>
<th>26%</th>
<th>84%</th>
<th>90%</th>
<th>95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>75</td>
<td>-77%</td>
<td>-68%</td>
<td>-47%</td>
<td>110%</td>
<td>171%</td>
<td>265%</td>
</tr>
<tr>
<td>500</td>
<td>-41%</td>
<td>-35%</td>
<td>-19%</td>
<td>38%</td>
<td>53%</td>
<td>74%</td>
</tr>
<tr>
<td>1000</td>
<td>-32%</td>
<td>-26%</td>
<td>-15%</td>
<td>24%</td>
<td>33%</td>
<td>45%</td>
</tr>
<tr>
<td>3000</td>
<td>-20%</td>
<td>-16%</td>
<td>-8%</td>
<td>14%</td>
<td>18%</td>
<td>24%</td>
</tr>
<tr>
<td>5000</td>
<td>-15%</td>
<td>-12%</td>
<td>-6%</td>
<td>11%</td>
<td>14%</td>
<td>19%</td>
</tr>
</tbody>
</table>
Table 11. GB2 VaR summary statistics (in millions) for 5000 bootstraps, true parameters $\theta = (0.837, 117516.887, 1.184, 1.454)$, and true VaR = 152m

<table>
<thead>
<tr>
<th>Samples</th>
<th>Mean VaR</th>
<th>Mean Bias</th>
<th>Median VaR</th>
<th>Median Bias</th>
<th>SD VaR</th>
<th>Relative SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>75 samples</td>
<td>1,203</td>
<td>690%</td>
<td>221</td>
<td>45.2%</td>
<td>5,183</td>
<td>3403%</td>
</tr>
<tr>
<td>500 samples</td>
<td>211</td>
<td>38.2%</td>
<td>144</td>
<td>-5.24%</td>
<td>216</td>
<td>142%</td>
</tr>
<tr>
<td>1000 samples</td>
<td>177</td>
<td>16.3%</td>
<td>146</td>
<td>-4.01%</td>
<td>120</td>
<td>78.8%</td>
</tr>
<tr>
<td>3000 samples</td>
<td>160</td>
<td>4.83%</td>
<td>150</td>
<td>-1.74%</td>
<td>58.8</td>
<td>38.6%</td>
</tr>
<tr>
<td>5000 samples</td>
<td>156</td>
<td>2.36%</td>
<td>150</td>
<td>-1.47%</td>
<td>44.8</td>
<td>29.4%</td>
</tr>
</tbody>
</table>

Table 12. GB2 VaR confidence bounds (relative difference) for 5000 bootstraps, true parameters $\theta = (0.837, 117516.887, 1.184, 1.454)$, and true VaR = 152m

<table>
<thead>
<tr>
<th>Samples</th>
<th>5%</th>
<th>10%</th>
<th>25%</th>
<th>84%</th>
<th>90%</th>
<th>95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>75 samples</td>
<td>-89%</td>
<td>-82%</td>
<td>-54%</td>
<td>814%</td>
<td>1450%</td>
<td>3154%</td>
</tr>
<tr>
<td>500 samples</td>
<td>-75%</td>
<td>-68%</td>
<td>-46%</td>
<td>127%</td>
<td>188%</td>
<td>289%</td>
</tr>
<tr>
<td>1000 samples</td>
<td>-66%</td>
<td>-57%</td>
<td>-36%</td>
<td>80%</td>
<td>116%</td>
<td>164%</td>
</tr>
<tr>
<td>3000 samples</td>
<td>-46%</td>
<td>-39%</td>
<td>-22%</td>
<td>41%</td>
<td>57%</td>
<td>77%</td>
</tr>
<tr>
<td>5000 samples</td>
<td>-38%</td>
<td>-31%</td>
<td>-18%</td>
<td>29%</td>
<td>41%</td>
<td>57%</td>
</tr>
</tbody>
</table>

4. Conclusions and Future Steps

The goals of the above analyses as stated in the introduction were to (1) assess asymptotic normality as an approximation for parameter estimation confidence intervals, and (2) guide model-development and validation for operational risk VaR models in the face of small sample-sizes. For the second question, the basic sub-questions that we set out to answer were (a) which heavy-tailed distributions lead to more stable OpVaR models for small-sample sizes and (b) how many losses are required for stable MLE parametric tail fitting.

Not surprisingly, approximating MLE parameter confidence intervals via asymptotic normality performs badly when the assumptions of asymptotic normality either are grievously violated (e.g. for the Weibull distributions considered here), or are questionable (e.g. for the GB2 distributions, for which the Fisher information matrix is possibly singular). For the remaining distributions, Theorem 2.1 appears to hold even for small sample-sizes, and the corresponding confidence intervals provide a good match with simulation (within 6%).

Turning to question (2), a trend emerges from our analyses with respect to stability:

GB2 < Pareto ≤ log-logistic < lognormal < Weibull*

The asterisk by the Weibull distribution is to warn the reader of the significant underestimation of the single-loss approximation. The absence of strict inequality between the Pareto and log-logistic distributions arises from both UOM1 and UOM2, where the relative standard deviations and confidence intervals for these distributions are comparable across the range of sample-sizes. It is interesting to note that the GB2 distribution fares worst even for UOM4, where it has bounded

---

1Recall that results for UOM2, UOM3, and UOM4 can be found in [Lar15]
mean and the Pareto distribution has unbounded mean (and true OpVaR more than 20x that of GB2!). There are several possible reasons for the instability of the GB2 distribution for OpVaR. The most obvious is the flexibility of the distribution. With four parameters to fit, the potential for over-fitting data is greater.

A second place to look is the Fisher information matrix. As seen by the plots in Sections 2.4 and Lar15, the wide variance for the parameter estimators matches the predictions of Theorem 2.1 for sample-sizes greater than 1500. While we have made only a passing reference asymptotic efficiency until now, the basic claim is that MLE provides the lowest variance of all asymptotically unbiased estimators. Applied here, asymptotic efficiency says that there is little room for improvement of the high instability of parameter fitting (and OpVaR estimation) for the GB2 distribution. Since the GB2 distribution nests the Weibull, lognormal and log-logistic distributions, we see that generality comes at a high price with respect to stability.

With respect to goodness-of-fit, the QQ-plots for the GB2 distribution looked best across the four loss data sets. The Weibull distribution tended to underestimate the highest losses (as did lognormal to a lesser degree), and the Pareto and log-logistic tended to overestimate the highest losses (Pareto dramatically so for UOM4). For all loss data sets, the log-logistic gave a poor fit the lower losses, resulting in overestimation. The poor fit is one likely reason for the instability of the log-logistic distribution seen here.

The question (b) feeds into other modeling decisions, such as determination of tail thresholds and clustering of data, i.e. can multiple Business Line/Event Type cells be merged into a single UOM for the purpose of MLE severity calibration to increase the sample-size. These decisions must also take into account data homogeneity considerations (with respect to clustering, see the paper Cop10), as well as goodness-of-fit. Both data homogeneity and goodness-of-fit should in general take precedence over stability considerations. More broadly, the question “how many losses are required” assumes that the modeler has sufficient methodological freedom.

Our analyses therefore provide more of a framework for modelers than a recipe. Such a framework could start with an OpVaR stability ‘appetite’ to develop criteria for each UOM (e.g. Business Line/Event Type cell, or collection of these). This stability appetite would also depend on which stability measure is used, e.g. the upper confidence bounds given above, or some other measure such as relative standard deviation. One example would be a ‘confidence-error’ rule: at the x% confidence level require a sufficient number of losses so that the relative error is not more than y%. For the lognormal distribution, a 90-30 rule would translate for UOM1 into requiring approximately 500 losses above the tail threshold (here set to 100k).

In the standard procedure of obtaining the total OpVaR by adding the OpVaR for each UOM, converting a stability rule per UOM to one for the whole model is relatively straightforward. If diversification benefits are considered, however, the this translation would be more involved. In both cases, however, it will likely be impossible to have one set of stability rules that can apply to all UOMs.

Even if some UOM-stability ‘breaches’ are allowed, it is entirely possible that certain banks (or sub-entities of a bank for which a stand-alone OR model is being requested) would pull out all of the methodological stops and still have an OpVaR model that exceeds any reasonable stability appetite. In such a situation, the
framework presented here can serve as statistical-backing for discussion with regulators and stakeholders about alternative to an MLE-based model. No one would propose an MLE-based LDA model for a bank or sub-entity having only a handful of internal losses, but when the numbers are in the hundreds of losses, developers and validators need statistical measures to determine when an MLE-based LDA model is inappropriate, and a fall-back solution must be sought.

By measuring stability over a range of loss data sets, we can also draw conclusions about each of the severity distributions across a range of risk profiles. We can order the heaviness of the various UOMs in terms of true VaR for a fixed distribution, mean, median, and kurtosis (in that order of importance):

\[ UOM_2 < UOM_1 < UOM_4 < UOM_3. \]

From the true OpVaR(99.9) for the different distributions, the difference between UOM1 and UOM4 is not as pronounced as the differences among the other UOMs. One a priori reasonable hypothesis would be that heavier loss data would require a larger sample-size to achieve the same stability as lighter data.

To check this intuition, we return to the stability-rule example above for the lognormal distribution. A 90-30 rule would correspond to 500 losses above 100k for UOM1, approximately 400 for UOM2, 500 for UOM4, and approximately 800 for UOM3. For the log-logistic distribution, the 90-30 rule would correspond to approximately 1000 losses for UOM1, UOM2 and UOM4, and 1500 losses for UOM3. For the Pareto distribution, the corresponding numbers are 1500 losses for UOM1, approximately 700 for UOM2, and more than 3000 losses for UOM4 (the SLA OpVaR(99.9) approximation for UOM3 differs significantly from the MC OpVar(99.9), so we pass over these results). There seems to be a trend that heavier loss data requires larger sample sizes for comparable levels of stability, but not a particularly strong one, at least based on the metrics used for heaviness of loss data here.

For the lognormal distribution, the relative constancy of a 90-30 stability rule across UOMs with true OpVar(99.9) values ranging from 14m to 495m is a significant advantage when it comes to applying these results to OpVaR models consisting of multiple units of measures, each with varying degrees of heaviness. In contrast, a model employing the Pareto distribution would require different numbers of losses to satisfy a given stability rule according to the underlying data.

While parametric bootstrapping is a natural way to measure stability of OpVaR models for the theory of MLE, in the general scheme of OpVaR models it gives a rather severe test of stability. A stability measure more related to quarter-to-quarter behavior would be the test mentioned in the introduction of inserting artificial losses while keeping the existing ones, rather than entirely swapping out the loss data, as occurs with parametric bootstrapping. It would be very interesting to study the relation between these two measures of stability. In particular, understanding how these notions of stability relate would facilitate defining a meaningful parametric bootstrapping stability rule, since we could calibrate the rule to the more intuitive quarter-on-quarter stability measure.

Although possible biases of MLE-based OpVaR models are not the focus of our investigations, we take a moment to discuss how our results relate to the claims of an “upward capital bias” for MLE-based OpVaR models in [OC12]. Section 3 confirms the general claim of a positive bias when comparing mean OpVaR of the

\[ \text{based on interpolation} \]
bootstrapped parameters to true OpVaR (see also the Appendix Lar15). Two qualifications should however be made. First, if our test statistic is the median bias rather than mean, then the bias is negative. Informally put, median bias indicates how often an MLE-based OpVaR model over or underestimates OpVaR, while mean bias takes into account the extent of the over or underestimation. From the risk management perspective, both quantities are meaningful, and having one negative and the other positive makes it difficult to claim that MLE leads to OpVaR overestimation.

Second, the authors of OC12 write that the mean OpVaR bias is a consequence of Jensen’s inequality, but no further details are given. This would follow if the CDF $F(x|\theta)$ for a heavy-tailed distribution were a convex function. There is no mention whether convexity is with respect to the loss variable $x$ or with respect to the parameters $\theta$. For the Jensen’s inequality argument of OC12 to be valid, convexity must be shown with respect to the parameters $\theta$, not the loss amount $x$. Specifically, we would have to show that, for all loss amounts $x$ in a neighborhood of the true OpVaR, the Hessian of $F(x|\theta)$ with respect to $\theta$ is negative definite (and hence the Hessian of the quantile function of $F(x|\theta)$ would be positive definite). This property is trivial to verify for the Pareto distribution considered here as depending only on one variable, but is less than straightforward for more complicated distributions.

We have restricted discussion thus far to MLE-based OpVaR modeling under assumptions required for MLE, such as the assumption of independent, identically distributed loss data, and the assumption that our model is correctly specified (hence the references to the “true” distribution). The applicability of these assumptions to OpVaR models is, however, a matter of debate in the literature. The articles She10, Erg08, OC12 address the failure of these assumptions to hold, and propose as alternatives to MLE, Bayesian inference, quantile-matching, and robust statistics, respectively.

The relatively high number of losses required to achieve a relative standard deviation of less than 10% (or the 90-30 rule mentioned above) for even the most stable distributions considered here provides further motivation to investigate alternative fitting techniques. In Erg08, quantile matching is shown to have desirable properties for operational risk data, even in the face of misspecified models. Bayesian inference and quantile-matching both give indications of better stability than MLE. It would be interesting to perform analogous OpVaR stability studies for these alternatives.

An obvious way to increase the stability of any OpVaR model—though one that is not allowed under current regulation—would be to reduce the required confidence level of 99.9% for regulatory capital. One proposal to preserve conservatism while potentially increasing stability is to calculate at a lower quantile (e.g. at 95%) and scale up by a multiplicative factor. It would be very interesting to compare the stability of such a scaled-up OpVaR to the results of this paper. A related question

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3The authors of OC12 likely intend to claim that $F(x|\theta)$ is convex with respect to $x$ for heavy-tailed distributions, but even this is not in general true. For a counterexample, let $F(x) = 1/(x(\sin(x)+1)+3) + 1/(x^2 + 3)$ for $x \geq 0$ and $1$ otherwise. Then $F(x)$ is a regularly varying—and hence subexponential—tail function, but has arbitrarily large inflection points, and hence is not convex even for large values of $x$. 

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is if such an approach would reduce the observed differences among the different severity distributions.

References


